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A DOUBLE APPROXIMATION OF FINITE DEFORMATIONS OF A SHELL

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A double approximation of the displacement field in the transverse direction is applied to model finite deformations of a shell. One (linear) approximation is used to compute the tangential derivatives of the field and the other (quadratic) approximation is used for the normal derivative. As a result, the field gradient is approximated by a linear function of the transverse coordinate, and the Green finite strain tensor is approximated by a quadratic function of the transverse coordinate. A two-dimensional model of shell deformation, which is consistent with the double approximation, contains three desired vectors as internal kinematic variables. However, only two of the vectors (the coefficients of the linear approximation) are parameters of the kinematic boundary conditions. They give six scalar degrees of freedom to a transverse fiber of the shell. The model constructed determines all components of the volume finite strain tensor and can be recommended for numerical analysis of deformation problems of shells that are nonuniform and layered in thickness.

The theory of shells that is based on a linear approximation of the displacement field along the normal coordinate is contradictory. It introduces an irremovable error into the physical (constitutive) equations for the material of the shell. The value of this error depends on the degree of anisotropy and inhomogeneity of the material. An attempt has been made [1, 2] to reduce the error by means of an additional scalar parameter that corrects the linear approximation of the displacement field. The concept of a double approximation realized below was developed under the influence of publications [3, 4]. It corrects and generalizes the results of [1, 2].

1. Deformation of a Shell Considered as a Cauchy Body. Let a shell-shaped body in its initial (unstressed) state occupy a region (volume) G. The material of the shell is distributed on a basic surface A. A curvilinear system of coordinates t_I with a local basis $\mathbf{a}_I$, in which $\mathbf{a}_1$ and $\mathbf{a}_2$ are tangential vectors, and $\mathbf{a}_3$ is a normal unit vector, is related to the basic surface. The position vector $\mathbf{g}(t_I)$ of an arbitrary point of the shell is given by the equality $\mathbf{g} = \mathbf{a} + t_3\mathbf{a}_3$, where $\mathbf{a}(t_i)$ is the position vector of a point of the basic surface. Here and below, the upper-case Latin subscripts take values 1, 2, and 3, and the lower-case subscripts take values 1 and 2; the tensor summation rule is used; ∂_I and ∇_I are the operators of partial and covariant differentiation with respect to the coordinate t_I; the possible dependence on time is not indicated explicitly.

The formula

$$\partial_I\mathbf{g} \equiv \mathbf{g}_I = \mathbf{a}_I + t_3\mathbf{b}_I$$

introduces the initial basis of the coordinate grid at an arbitrary point of the shell. In accordance with the definition of the basis, the equalities $\mathbf{a}_i \equiv \partial_i\mathbf{a}$, $\mathbf{a}_i \cdot \mathbf{a}_3 \equiv 0$, $\mathbf{b}_i \equiv \partial_i\mathbf{a}_3$, $\mathbf{b}_i \cdot \mathbf{a}_3 \equiv 0$, and $\mathbf{b}_3 \equiv 0$ are valid.

The shell surface usually consists of two outer surfaces A_n and an edge surface A_3, which is orthogonal to the basic surface along its boundary contour C. The surfaces A and A_n are given by the equalities $t_3 = 0$ and $t_3 = h_n$ so that $h_1 \leqslant t_3 \leqslant h_2$ (h_1 and h_2 are functions of the surface point or constant numbers). Each surface A_N is oriented by the normal unit vector $\mathbf{e}_N$.

The differentials of the volume and surfaces are determined by the equalities $dG \equiv J dt_3 dA$, $dA \equiv a dt_1 dt_2$, $dA_n \equiv j_n dA$, and $dA_3 \equiv j_3 dt_3 dC$, in which $aJ(\mathbf{g})$ is the Jacobian of the curvilinear system relative to the Cartesian system; $j_N(\mathbf{g} \in A_N)$ are the metric parameters of the surfaces (j_n does not depend on t_3).

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Deformation of the shell transforms the initial position vector into the instantaneous vector $\mathbf{g}^+ = \mathbf{g} + \mathbf{w}$, where $\mathbf{w}(\mathbf{g})$ is the finite-displacement vector of an arbitrary material point of the shell. The initial basis $\mathbf{g}_I$ is transformed into the instantaneous basis

$$\mathbf{g}_I^+ \equiv \partial_I \mathbf{g}^+ = \mathbf{g}_I + \mathbf{w}_I, \quad \mathbf{w}_I \equiv \partial_I \mathbf{w}. \tag{1.1}$$

The vectors $\mathbf{w}_I(\mathbf{g})$ form the tensor gradient of the displacement field.

In the process of deformation, the shell is subjected to external mechanical actions, that is, surface and volume force fields. The volume field of the external forces is given by the density vector $\mathbf{f}(\mathbf{g})$ per initial unit volume. The surface field is divided into three fields, which are given at the edge and outer surfaces by the density vectors $\mathbf{f}_N(\mathbf{g} \in A_N)$ per initial unit area. Deformation of the shell as a Cauchy body generates a volume field of internal stresses, which is represented by contravariant Piola vectors $\mathbf{z}^I(\mathbf{g})$.

The balance of the external and internal forces acting on the shell can be expressed by the equation of virtual work ("weak" formulation):

$$\int_A \int_{h_1}^{h_2} (\mathbf{f} \cdot \delta \mathbf{w} - \delta z) J dt_3 dA + \int_A \mathbf{f}_n \cdot \delta \mathbf{w}_{(n)} j_n dA + \int_C \int_{h_1}^{h_2} \mathbf{f}_3 \cdot \delta \mathbf{w} j_3 dt_3 dC = 0. \tag{1.2}$$

Here $\delta \mathbf{w}$ is the virtual displacement vector; $\delta \mathbf{w}_{(n)}$ is its value on the surface A_n; δz is the density of the virtual strain energy per initial unit volume determined by any one of the equalities

$$\delta z = \mathbf{z}^I \cdot \delta \mathbf{w}_I = z^{IJ} \delta w_{IJ} = Z^{IJ} \delta W_{IJ}, \tag{1.3}$$

in which Z^{IJ} and z^{IJ} are components of the symmetric and asymmetric Piola stress tensors:

$$Z^{IJ} \equiv \mathbf{z}^I \cdot \mathbf{g}_+^J, \quad z^{IJ} \equiv \mathbf{z}^I \cdot \mathbf{a}^J; \tag{1.4}$$

w_{IJ} and W_{IJ} are components of the Green strain gradient and tensor:

$$w_{IJ} \equiv \mathbf{w}_I \cdot \mathbf{a}_J, \quad W_{IJ} \equiv \mathbf{g}_I^+ \cdot \mathbf{g}_J^+ - \mathbf{g}_I \cdot \mathbf{g}_J. \tag{1.5}$$

Variational equation (1.2) must be supplemented by constitutive equations (restrictions) for the material. In particular, purely mechanical elastic and elastoplastic deformations of many materials are governed by incremental constitutive relations of the form

$$\delta Z^{IJ} = E^{IJKL} \delta W_{KL}. \tag{1.6}$$

Their coefficients can contain information on the prehistory of loading of the body.

System (1.1)–(1.6) gives a weak formulation of the finite deformation problem of a shell as a Cauchy body.

2. An Approximate Model for Shell Deformation. In constructing an approximate model for finite deformation of a shell, a two-fold approximation of the displacement field is used:

$$\mathbf{w} \simeq \mathbf{w}^{(1)} \equiv \mathbf{u} + t_3 \mathbf{v}; \tag{2.1}$$

$$\mathbf{w} \simeq \mathbf{w}^{(2)} \equiv \mathbf{u} + t_3 \mathbf{u}_3 + (1/2)(t_3)^2 \mathbf{v}_3, \quad \mathbf{u}_3 \equiv \mathbf{v}, \tag{2.2}$$

where $\mathbf{u}(\mathbf{a})$, $\mathbf{v}(\mathbf{a})$, and $\mathbf{v}_3(\mathbf{a})$ are the desired kinematic vectors defined on the basic surface of the shell. Approximation (2.1) corresponds to the assumption of straight normals and is used to calculate the virtual displacement

$$\delta \mathbf{w} \simeq \delta \mathbf{w}^{(1)} = \delta \mathbf{u} + t_3 \delta \mathbf{v} \tag{2.3}$$

and the derivatives of the vectors

$$\mathbf{w}_i \simeq \partial_i \mathbf{w}^{(1)} = \mathbf{u}_i + t_3 \mathbf{v}_i, \quad \mathbf{u}_i \equiv \partial_i \mathbf{u}, \quad \mathbf{v}_i \equiv \partial_i \mathbf{v}. \tag{2.4}$$

Approximation (2.2) admits bending of transverse fibers and is used in computation of the vector

$$\mathbf{w}_3 \simeq \partial_3 \mathbf{w}^{(2)} = \mathbf{u}_3 + t_3 \mathbf{v}_3, \quad \mathbf{u}_3 \equiv \mathbf{v}. \tag{2.5}$$

Substitution of approximations (2.3)–(2.5) into the equation of virtual work (1.2) leads to its formulation in the two-dimensional space of the basic surface:

$$\int_A (\mathbf{p}\cdot\delta\mathbf{u}+\mathbf{q}\cdot\delta\mathbf{v}-\delta Z)dA+\int_C (\mathbf{p}_3\cdot\delta\mathbf{u}+\mathbf{q}_3\cdot\delta\mathbf{v})dC=0. \tag{2.6}$$

Here, we introduce the generalized vectors of the external forces,

$$\mathbf{p}\equiv\mathbf{f}_n j_n+\int_{h_1}^{h_2}\mathbf{f}J dt_3,\quad \mathbf{p}_3\equiv\int_{h_1}^{h_2}\mathbf{f}_3 j_3 dt_3,\quad \mathbf{q}\equiv\mathbf{f}_n j_n h_n+\int_{h_1}^{h_2}\mathbf{f}J t_3 dt_3,\quad \mathbf{q}_3\equiv\int_{h_1}^{h_2}\mathbf{f}_3 j_3 t_3 dt_3, \tag{2.7}$$

and the surface density of the virtual strain energy,

$$\delta Z\equiv\int_{h_1}^{h_2}\delta z J dt_3. \tag{2.8}$$

Use of the first equality from (1.3) leads to the formula

$$\delta Z=\mathbf{x}^I\cdot\delta\mathbf{u}_I+\mathbf{y}^I\cdot\delta\mathbf{v}_I, \tag{2.9}$$

in which the vectors

$$\mathbf{x}^I\equiv\int_{h_1}^{h_2}\mathbf{z}^I J dt_3,\quad \mathbf{y}^I\equiv\int_{h_1}^{h_2}\mathbf{z}^I J t_3 dt_3 \tag{2.10}$$

have the meaning of generalized internal forces: stresses and moments.

Substitution of (2.9) into (2.6) and integration by parts using equalities (2.4) yields the equation

$$\int_C [(\mathbf{p}_3-e_{3i}\mathbf{x}^i)\cdot\delta\mathbf{u}+(\mathbf{q}_3-e_{3i}\mathbf{y}^i)\cdot\delta\mathbf{v}]\,dC$$

$$+\int_A [(\mathbf{p}+\nabla_i\mathbf{x}^i)\cdot\delta\mathbf{u}+(\mathbf{q}-\mathbf{x}^3+\nabla_i\mathbf{y}^i)\cdot\delta\mathbf{v}-\mathbf{y}^3\cdot\delta\mathbf{v}_3]\,dA=0. \tag{2.11}$$

Since the variations $\delta\mathbf{u}$, $\delta\mathbf{v}$, and $\delta\mathbf{v}_3$ are independent, from (2.11) follow the local equilibrium equations

$$\nabla_i\mathbf{x}^i+\mathbf{p}=0,\quad \nabla_i\mathbf{y}^i-\mathbf{x}^3+\mathbf{q}=0,\quad \mathbf{y}^3=0, \tag{2.12}$$

which are defined on the basic surface and the force boundary conditions

$$e_{3i}\mathbf{x}^i=\mathbf{p}_3,\quad e_{3i}\mathbf{y}^i=\mathbf{q}_3 \tag{2.13}$$

on the section of the contour on which the forces are specified. On an attached section, the kinematic vectors $\mathbf{u}$ and $\mathbf{v}$ must be specified.

System (2.12) of static equations must be supplemented by constitutive relations, for example, of the form of (1.6). Then, equalities (2.10) will have the meaning of constitutive equations for generalized internal forces. The volume strain tensor must be expressed beforehand in terms of the desired kinematic parameters of the basic surface. For this, from (1.1), (2.4), and (2.5), we determine the vectors

$$\mathbf{g}_I^+=\mathbf{a}_I^++t_3\mathbf{b}_I^+,\qquad \mathbf{a}_I^+\equiv\mathbf{a}_I+\mathbf{u}_I,\qquad \mathbf{b}_I^+\equiv\mathbf{b}_I+\mathbf{v}_I \tag{2.14}$$

and then, from (1.5) we have

$$W_{IJ}=U_{IJ}+t_3V_{IJ}+(t_3)^2V_{IJ}^{(2)},\quad 2U_{IJ}\equiv\mathbf{a}_I^+\cdot\mathbf{a}_J^+-\mathbf{a}_I\cdot\mathbf{a}_J,$$
$$2V_{IJ}\equiv\mathbf{b}_I^+\cdot\mathbf{a}_J^++\mathbf{b}_J^+\cdot\mathbf{a}_I^+-\mathbf{b}_I\cdot\mathbf{a}_J-\mathbf{b}_J\cdot\mathbf{a}_I,\quad 2V_{IJ}^{(2)}\equiv\mathbf{b}_I^+\cdot\mathbf{b}_J^+-\mathbf{b}_I\cdot\mathbf{b}_J. \tag{2.15}$$

The quantities U_{IJ}, V_{IJ}, and $V_{IJ}^{(2)}$ are the two-dimensional parameters of shell deformation, which can be expressed, by means of (2.14), in terms of the surface kinematic vectors u_I and v_I.

A scalar formulation of the resulting relations is realized by means of expansion of all the kinematic and force vectors in terms of the initial basis:

$$\mathbf{u} = u_J\mathbf{a}^J, \quad \mathbf{v} = v_J\mathbf{a}^J, \quad \mathbf{u}_I = u_{IJ}\mathbf{a}^J, \quad \mathbf{v}_I = v_{IJ}\mathbf{a}^J, \quad \mathbf{x}^I = x^{IJ}\mathbf{a}_J,$$
$$\mathbf{p} = p^J\mathbf{a}_J, \quad \mathbf{p}_3 = p_3^J\mathbf{a}_J, \quad \mathbf{y}^I = y^{IJ}\mathbf{a}_J, \quad \mathbf{q} = q^J\mathbf{a}_J, \quad \mathbf{q}_3 = q_3^J\mathbf{a}_J. \tag{2.16}$$

In particular, the virtual equation (2.6) takes the form

$$\int_C (p_3^J\delta u_J + q_3^J\delta v_J)\,dC + \int_A (p^J\delta u_J + q^J\delta v_J - x^{IJ}\delta u_{IJ} - y^{IJ}\delta v_{IJ})\,dA = 0 \tag{2.17}$$

with the kinematic variables

$$u_{3J} \equiv v_J, \quad u_{iJ} \equiv \mathbf{a}_J\cdot\partial_i\mathbf{u}, \quad v_{iJ} \equiv \mathbf{a}_J\cdot\partial_i\mathbf{v}, \tag{2.18}$$

and with the static variables x^{IJ} and y^{IJ}, which are asymmetric stresses and moments. From (2.10) follow the equalities

$$x^I_{\cdot J} = X^{IM}a^+_{MJ} + Y^{IM}b^+_{MJ}, \quad y^I_{\cdot J} = Y^{IM}a^+_{MJ} + Y^{IM}_{(2)}b^+_{MJ}, \tag{2.19}$$

which determine the asymmetric stresses and moments in terms of symmetric stresses and moments,

$$X^{IJ} \equiv \int_{h_1}^{h_2} Z^{IJ}J\,dt_3, \quad Y^{IJ} \equiv \int_{h_1}^{h_2} Z^{IJ}Jt_3\,dt_3, \quad Y^{IJ}_{(2)} \equiv \int_{h_1}^{h_2} Z^{IJ}J(t_3)^2\,dt_3, \tag{2.20}$$

and the kinematic parameters

$$a^+_{IJ} \equiv \mathbf{a}^+_I\cdot\mathbf{a}_J = a_{IJ} + u_{IJ}, \; a_{IJ} \equiv \mathbf{a}_I\cdot\mathbf{a}_J, \; b^+_{IJ} \equiv \mathbf{b}^+_I\cdot\mathbf{a}_J = b_{IJ} + v_{IJ}, \; b_{IJ} \equiv \mathbf{b}_I\cdot\mathbf{a}_J. \tag{2.21}$$

Equalities (2.19), together with (1.6), (2.15), (2.20), and (2.21), formulate generalized constitutive relations for asymmetric stresses and moments.

System (2.7), (2.10), (2.12), (2.13), and (2.18) formulates a two-dimensional problem of shell deformation, which is consistent with double approximation (2.1) and (2.2) of the displacement field. The weak equation (2.6) with (2.9) or its scalar formulation (2.17) are preferable to the local static equations (2.12) and (2.13) for numerical analysis.

As a result of solving the two-dimensional problem, the following kinematic and static variables are determined: the vector fields of displacement u_J, v_J, and v_{3J}, the tensor fields of strains U_{IJ}, V_{IJ}, and $V^{(2)}_{IJ}$ and of the generalized stresses x^{IJ} and y^{IJ}. The ultimate goal of the shell deformation problem is to determine the volume fields of displacements, strains, and stresses. The displacement and strain fields inside the shell are computed by formulas (2.2) and (2.15) and the components of the stress vectors $\mathbf{z}^I$ are computed by means of local constitutive relations of the form (1.6) or others. The stress vector $\mathbf{z}^3$ and its components are found by integration of the static Cauchy equation

$$\partial_3(J\mathbf{z}^3) = -\partial_i(J\mathbf{z}^i) - J\mathbf{f} \tag{2.22}$$

with the already known vectors $\mathbf{z}^i$. In the integration of (2.22), the force conditions at one of the outer surfaces of the shell are used, and the conditions at the other surface are satisfied owing to the generalized equations (2.12). For layered shells, static conditions at the interlayer surfaces must be satisfied.

The model of finite deformation of a shell with double approximation of the field of finite displacements contains a closed formulation of the two-dimensional boundary-value problem, and three-dimensional relations for reconstruction of the volume fields of displacements, strains, and stresses. The last equation from the generalized static equations (2.12) is absent from the traditional formulation, which uses a linear approximation of the displacement field of the shell. This equation, using constitutive relations, makes it possible to exclude the displacement vector $\mathbf{v}_3$ from the set of unknown kinematic variables. Only two displacement vectors ($\mathbf{u}$ and $\mathbf{v}$) are the kinematic parameters of the generalized model, since only these are the parameters of the kinematic boundary conditions. Their components form six local degrees of freedom of the deformable shell. The resulting generalized model differs from the models given in [1, 2], since it introduces the additional kinematic vector $\mathbf{v}_3$ instead of the scalar parameter V_{33}, and, correspondingly, the model proposed here contains the additional vector equation $\mathbf{y}^3 = 0$ instead of the scalar equality $Y^{33} = 0$. The greatest difference between the models compared should be expected for shells that are nonuniform and layered in thickness.

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